ALGEBRAIC THEORY OF THE EXPRESSIBILITY OF CUBIC FORMS AS DETERMINANTS, WITH APPLICATION TO DIOPHANTINE ANALYSIS.

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1. It was proved geometrically by H. Schröter* and more simply by L. Cremona† that a sufficiently general cubic surface $f = 0$ is the locus of the intersections of corresponding planes of three projective bundles of planes:

$$\kappa l_{11} + \lambda l_{12} + \mu l_{13} = 0, \qquad \kappa l_{21} + \lambda l_{22} + \mu l_{23} = 0, \qquad \kappa l_{31} + \lambda l_{32} + \mu l_{33} = 0,$$

where κ, λ, μ are parameters and the l_{ij} are linear homogeneous functions of $x_1, \cdots, x_4$. Hence the surface is expressible in determinantal form $|l_{ij}| = 0$. Solving the three equations, we get

$$x_1 : x_2 : x_3 : x_4 = f_1 : f_2 : f_3 : f_4,$$

where f_i is a homogeneous cubic function of κ, λ, μ. We thus secure a parametric representation of the points of the surface, which is therefore unicursal.

In case all of the coefficients of the l_{ij} are rational, we have the complete solution in rational numbers of the Diophantine equation $f = 0$. However, there exist cubic equations $f = 0$ whose rational solutions involve three parameters homogeneously such that f is not expressible rationally in determinantal form (§ 12).

With the application to Diophantine analysis in mind, I here discuss algebraically the problem to express a given cubic form f as a determinant $|l_{ij}|$ of the third order. Since we are interested ultimately in the case in which the coefficients of the l_{ij} are rational and since three linear functions of four variables vanish for values not all zero of the variables, we shall assume that f vanishes at a known rational point. Then the coefficients of the l_{ij} are shown to be expressible rationally in terms of a root of an algebraic equation whose leading coefficient is not zero if $f = 0$ has no singular point. The existence of a rational root may be decided by a finite number of trials.

The corresponding binary problem is solved by the identity

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* *Journal für Mathematik*, Vol. 62 (1863), p. 265.

† *Ibid.*, Vol. 68 (1868), p. 79. Cf. Clebsch, *ibid.*, 65 (1866), p. 359.

$$ax^3 + bx^2y + cxy^2 + dy^3 \equiv \begin{vmatrix} ax + by & y & 0 \\ -cy & x & y \\ dy & 0 & x \end{vmatrix}.$$

In the ternary case we obtain simple geometrical criteria that a cubic curve with a rational point be expressible rationally in determinantal form. There exist determinants of the third order which vanish for no rational point, for example, the determinant $\Delta(\xi)$ of the general number $\xi = x + y\theta + z\zeta$ of the algebra having the units 1, θ, $\zeta = \theta^2$, where θ is a root of a cubic with no rational root. But ternary forms without rational solutions have little interest in Diophantine analysis and will not be treated here.

Given one representation of a form as a determinant, we may derive an infinitude of representations by the familiar operations which leave a determinant unaltered in value. The more definitive problem treated here is to find a representative of each class of equivalent matrices with linear elements which have a given determinant. This raises the question of the number of classes of such matrices.

In another paper* I prove that every binary form, every ternary form, and every quaternary quadratic form are expressible in determinantal form; while, apart from these and the quaternary cubic, no further general form has this property. The rarety of such forms justifies the present investigation of cubic forms with attention to rationality. The theory covers every cubic form, not merely a sufficiently general one.

2. The given rational point on the locus can be transformed rationally into $(1, 0, \cdots, 0)$. Then the locus is $f = 0$, where $f = x^2f_1 + xf_2 + f_3$, where f_j is of degree j in $y, z, \cdots$. If f_1 is not identically zero, we take it as the new variable y. Then by adding to x a suitable linear function of $y, z, \cdots$, we may delete rationally from f_2 all terms with the factor y. Hence

$$f = x^2y + xf_2 + f_3, \tag{1}$$

where the quadratic f_2 is free of x and y, and the cubic f_3 is free of x. But if $f_1 \equiv 0$ and $f_2 \not\equiv 0$, we may transform f_2 into $ay^2 + q$, where $a \neq 0$ and q lacks y. By adding to x a suitable linear function of $y, z, \cdots$, we may delete rationally from f_3 the terms with the factor y^2 and obtain $f = x(ay^2 + q) + yr + s$, where q, r, s lack both x and y. Taking ax as a new x, we have $a = 1$. Interchanging x and y, we obtain a form of type (1).

3. Let the form (1) equal to a determinant of the third order whose elements are linear forms with rational coefficients. We may assume that x occurs in the first element of the first row and that its coefficient is unity.

* *Transactions Amer. Math. Soc.*, April, 1921.

The remaining elements in the first row and first column may be assumed to be free of x. By interchanges of the last two rows and last two columns, we may assume that the second element of the second row contains x; its coefficient may be made equal to unity. Hence we may take

$$f = \begin{vmatrix} x + l_1 & l_2 & l_3 \\ l_4 & x + l_5 & l_6 \\ l_7 & l_8 & y \end{vmatrix}, \tag{2}$$

where each l_i lacks x. After subtracting multiples of the last row and last column from the remaining rows and columns, respectively, we may assume that l_3, l_6, l_7, l_8 lack x and y. Thus by the terms linear in x,

$$l_1 + l_5 = 0, \qquad - l_3 l_7 - l_6 l_8 = f_2. \tag{3}$$

The interchange of the first two columns and the first two rows of (2) corresponds to the substitution

$$S = (l_1 l_5)(l_2 l_4)(l_3 l_6)(l_7 l_8).$$

The interchange of rows with columns corresponds to

$$T = (l_2 l_4)(l_3 l_7)(l_6 l_8).$$

Add the products of the elements of the second row by k to the elements of the first row and then subtract the products of the elements of the first column by k from the elements of the second column; we obtain a determinant of the same form with the same l_4, l_6, l_7 and with

$$\begin{gathered} l_1' = l_1 + kl_4, \qquad l_2' = l_2 - kl_1 + kl_5 - k^2 l_4, \qquad l_5' = l_5 - kl_4, \\ l_3' = l_3 + kl_6, \qquad l_8' = l_8 - kl_7. \end{gathered} \tag{4}$$

Proceeding as before with the words first and second interchanged, we obtain a like determinant with the same l_2, l_3, l_8 and with

$$\begin{gathered} l_1' = l_1 - kl_2, \qquad l_4' = l_4 + kl_1 - kl_5 - k^2 l_2, \qquad l_5' = l_5 + kl_2, \\ l_6' = l_6 + kl_3, \qquad l_7' = l_7 - kl_8. \end{gathered} \tag{5}$$

The last result may be obtained by transforming (4) by S.

4. For 3 variables (1) becomes

$$f = x^2 y + exz^2 + C, \qquad C = \alpha y^3 + \beta y^2 z + \gamma y z^2 + \delta z^3. \tag{6}$$

The coefficient of x^3 in the Hessian of (6) is $-8e$. Hence $(1, 0, 0)$ is a point of inflexion of $f = 0$ if and only if $e = 0$. If $e \neq 0$, we multiply y by e^2 and x by $1/e$ and obtain a form of type (6) with $e = 1$. We seek the conditions under which (6) can be expressed as a determinant (2), where l_3, l_6, l_7, l_8 are multiples of z.

First, let either $l_3 \equiv l_6 \equiv 0$ or $l_7 \equiv l_8 \equiv 0$. Then (2) is the product of y by a quadratic which evidently vanishes when $x = -l_1$, $l_2 = 0$. Hence (6) must be the product of y by a ternary quadratic q which vanishes at a rational point. Conversely, the product of such a q by a rational linear form l can be transformed* rationally into one of

$$l(xy - kz^2) = \begin{vmatrix} l & 0 & 0 \\ 0 & x & z \\ 0 & kz & y \end{vmatrix}, \qquad l(ay^2 + byz + cz^2) = \begin{vmatrix} l & 0 & 0 \\ 0 & ay + bz & z \\ 0 & -cz & y \end{vmatrix}.$$

However, the product of a linear form, say y, by an arbitrary ternary quadratic form Q is expressible rationally as a determinant. If Q lacks x^2, it vanishes at (1, 0, 0). In the contrary case we may delete the terms in xy and xz by a transformation on x and use identity (7).

Second, let l_3, l_6, l_7, l_8 be not all zero identically. Since S, T and ST replace l_7 by l_8, l_3, l_6, we may take $l_7 \not\equiv 0$. Hence we may set $l_7 = z$. By (4), we may take $l_8 = 0$. By (3), $-l_3 z \equiv ez^2$, whence $l_3 = -ez$.

We first treat the case $e = 0$. Then $l_3 = l_8 = 0$, $l_7 = z$. Hence shall

$$C = \begin{vmatrix} l_1 & l_2 & 0 \\ l_4 & -l_1 & l_6 \\ z & 0 & y \end{vmatrix},$$

where l_6 is a multiple bz of z. If $l_2 = ay$, $C = y(-l_1^2 - ayl_4 + abz^2)$, which requires $\delta = 0$ in C and is then satisfied if $a = 1$, $l_1 \equiv 0$, $b = \gamma$, $l_4 = -\alpha y - \beta z$, so that we have the identity†

$$(7) \qquad y(x^2 + \alpha y^2 + \beta yz + \gamma z^2) = \begin{vmatrix} x & y & 0 \\ -\alpha y - \beta z & x & \gamma z \\ z & 0 & y \end{vmatrix}.$$

Next, let l_2 contain z, whose coefficient may be divided out of the second column of determinant C and multiplied into its second row. Then $l_2 = z + cy$. By (5), we may delete z from l_1 and set $l_1 = ay$. Write $l_4 = gy + hz$, $l_6 = tz$. Then shall

$$C = \begin{vmatrix} ay & z + cy & 0 \\ gy + hz & -ay & tz \\ z & 0 & y \end{vmatrix}$$
$$= -(a^2 + cg)y^3 - (g + ch)y^2 z + (ct - h)yz^2 + tz^3,$$

whence

$$t = \delta, \qquad h = c\delta - \gamma, \qquad g = -ch - \beta, \qquad a^2 + cg = -\alpha.$$

The last relation furnishes the condition

* Dickson, "Algebraic Invariants," 1914, p. 24.

† If (6) with $e = 0$ has any linear factor, it has the factor y.

$$(8) \qquad -\alpha + \beta c - \gamma c^2 + \delta c^3 = a^2,$$

which is solvable in rational numbers if and only if (6) with $e = 0$ contains the rational point* $(a, 1, -c)$. The latter may be any point not on the inflexion tangent $y = 0$; if $\delta \neq 0$, it is any point except $(1, 0, 0)$.

Next, let $e = 1$, whence $l_3 = -z$. We have $l_6 = kz$. By (5), we may take $l_6 \equiv 0$. Thus (2) becomes

$$(9) \qquad \begin{vmatrix} x + l_1 & l_2 & -z \\ l_4 & x - l_1 & 0 \\ z & 0 & y \end{vmatrix} = x^2y + xz^2 - y(l_1^2 + l_2 l_4) - z^2 l_1.$$

Identify this with (6). By the terms in z^3, we see that $l_1 = ay - \delta z$. Set $l_2 = cy + dz$, $l_4 = gy + hz$. Then the conditions are

$$cg = -\alpha - a^2, \qquad dg + ch = 2a\delta - \beta, \qquad dh = -\gamma - \delta^2 - a.$$

Also, $dg - ch = 2r$ must be rational. From

$$(dg - ch)^2 \equiv (dg + ch)^2 - 4cg \cdot dh,$$

we obtain the condition that

$$(10) \qquad -4a^3 - 4a^2\gamma - 4a(\alpha + \beta\delta) + \beta^2 - 4\alpha(\gamma + \delta^2) = 4r^2$$

shall have rational solutions a, r. Then we have rational values of $dg = A$, $ch = B$, $dh = C$, $cg = D$ such that $AB = CD$. If $A \neq 0$ or $C \neq 0$, we may take $d = 1$, $g = A$, $h = C$, $c = D/A$ or B/C. If $B \neq 0$ or $D \neq 0$, we may take $c = 1$, $g = D$, $h = B$, $d = C/B$ or A/D.

To interpret (10), replace x by $l_1 = ay - \delta z$ in (6) with $e = 1$, and remove the factor y. We obtain a quadratic in $y : z$ with a rational root† if and only if (10) has rational solutions. Then (6) has three rational points of intersection with $x = ay - \delta z$ for some rational value of a, i.e., for some line other than $y = 0$ through $(-\delta, 0, 1)$. The latter point is the tangential of $P = (1, 0, 0)$, i.e., the new point in which the tangent $y = 0$ at P to the cubic curve meets the curve.

THEOREM. *Any reducible cubic curve is expressible rationally in determinantal form. An irreducible cubic curve with a rational inflexion point is expressible rationally in determinantal form if and only if it contains a further rational point. A cubic curve with a rational point P not an inflexion is expressible rationally in determinantal form if and only if it has three rational points of intersection with some line, other than the tangent at P, through the tangential of P.*

5. For four variables, we may set $f_2 = az^2 + bw^2$ in (1), where $b = 0$

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* For which $x = l_1$, $l_2 = 0$, whence the elements of the second column of (2) are all zero.

† Which makes $l_2 = 0$ in accord with the second column of (9).

if $a = 0$. First, let l_3, l_6, l_7, l_8 be not all free of z, the contrary case being treated in § 9. In view of substitutions S and T of § 3, we may assume that l_7 contains z; its coefficient may be made unity by removal of factors from the last row and last column of (2). By (4), we may assume that $l_8 = vw$. We here take $v \neq 0$ (treating $v = 0$ in § 10). By dividing the elements of the second column by v and multiplying those of the second row by v, we have $l_8 = w$. Then by (5) we may take $l_7 = z$. Hence,* by (3),

$$l_3 = -az + gw, \qquad l_6 = -gz - bw,$$

$$(11)\quad f = \begin{vmatrix} x + l_1 & l_2 & l_3 \\ l_4 & x - l_1 & l_6 \\ z & w & y \end{vmatrix} = x^2y + x(az^2 + bw^2) + \begin{vmatrix} l_1 & l_2 & l_3 \\ l_4 & -l_1 & l_6 \\ z & w & y \end{vmatrix}.$$

Set

$$(12)\quad l_1 = cy + dz + ew, \quad l_2 = hy + jz + kw, \quad l_4 = my + nz + pw,$$

$$(13)\quad f_3 = Ay^3 + By^2z + Cy^2w + Dyz^2 + Eyzw + Fyw^2 + Gz^3 + Hz^2w + Jzw^2 + Lw^3.$$

Then (11) is identical with (1) if and only if

$$(14)\quad \begin{gathered} -c^2 - hm = A, \quad -2cd - hn - jm = B, \quad -2ce - hp - km = C, \\ -d^2 - jn - gh - ac = D, \quad -2de - jp - kn + 2gc - am - bh = E, \\ -e^2 - kp + bc + gm = F, \qquad -ad - gj = G, \\ 2gd - an - bj - gk - ae = H, \\ bd + gn + 2ge - ap - bk = J, \qquad be + gp = L. \end{gathered}$$

We shall designate these equations as (A), $\cdots$, (L).

First, let $a = b = 0$. If $g = 0$, then $l_3 \equiv l_6 \equiv 0$ and (11) is the product of y by its minor. Postponing this case to the end of § 11, we have $g \neq 0$. Let $dg = \delta$, $eg = \epsilon$. Then equations (L), $\cdots$, (D) give

$$gp = L, \quad gj = -G, \quad gn = J - 2\epsilon, \quad gk = 2\delta - H, \quad g^3m = \epsilon^2 + g^2F + L(2\delta - H),$$

$$g^3h = -\delta^2 - g^2D + G(J - 2\epsilon), \qquad 2g^3c = g^2E - LG - HJ + 2\epsilon H + 2\delta J - 2\delta\epsilon.$$

Inserting these values into the product of (B) and (C) by g^4, we see that the terms of the third degree in δ and ϵ cancel, giving

$$(15)\quad \delta^2J + 2\delta\epsilon H + 3\epsilon^2G + \delta(g^2E - 3LG - HJ) + \epsilon(2g^2D - 4GJ) + g^4B - g^2(FG + DJ) + GJ^2 + GHL = 0,$$

$$(16)\quad 3\delta^2L + 2\delta\epsilon J + \epsilon^2H + \delta(2g^2F - 4HL) + \epsilon(g^2E - 3GL - HJ) + g^4C - g^2(FH + DL) + H^2L + GJL = 0.$$

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* It is proved in § 13 that no further normalization of our determinant is possible and that the two excluded cases are truly exceptional.

Multiplying (A) by $3g^6$, we obtain an equation whose terms of highest degree are $-6\delta^3L - 6\delta^2\epsilon J - 6\delta\epsilon^2H - 6\epsilon^3G$. Hence by adding to it the product of (15) by 2ϵ and the product of (16) by 2δ, we obtain the quadratic

$$
\begin{aligned}
&\delta^2(g^2F-5HL+3J^2)+\epsilon^2(g^2D-5GJ+3H^2)+\delta\epsilon(g^2E-21GL+5HJ)\\
&+\delta[2g^4C+g^2(3EJ-2FH-8DL)+5GJL+2H^2L-3J^2H]\\
&+\epsilon[2g^4B+g^2(3EH-2DJ-8FG)+5GHL+2GJ^2-3H^2J]\\
&+3g^6A+\tfrac{3}{4}(g^2E-LG-HJ)^2-3(g^2D-GJ)(g^2F-HL)=0.
\end{aligned}
\tag{17}
$$

The true resultant of three ternary quadratic forms u, v, w in x, y, z is known* to be the determinant of the coefficients of $x^2, xy, xz, y^2, yz, z^2$ in u, v, w, j_x, j_y, j_z, where j is the Jacobian (functional determinant) of u, v, w, while j_x is its partial derivative with respect to x. The resultant is of the fourth degree in the coefficients of each form u, v, w. To find the degree in g of the resultant R of our forms (15)–(17), we shall determine the terms of R of highest degree in g. Writing the functions (15)–(17) homogeneously in δ, ϵ, τ and retaining in each coefficient only the term of highest degree in g, we have

$$
\begin{gathered}
\delta^2J + 2\delta\epsilon H + 3\epsilon^2G + \delta\tau g^2E + 2\epsilon\tau g^2D + \tau^2g^4B,\\
3\delta^2L + 2\delta\epsilon J + \epsilon^2H + 2\delta\tau g^2F + \epsilon\tau g^2E + \tau^2g^4C,\\
g^2(\delta^2F + \delta\epsilon E + \epsilon^2D + 2\delta\tau g^2C + 2\epsilon\tau g^2B + 3\tau^2g^4A).
\end{gathered}
$$

Thus $R = (g^2)^4r$, where r is the resultant when the factor g^2 is omitted from the last form. We make the transformation of variables $\delta = w$, $\epsilon = z$, $g^2\tau = y$ and obtain the partial derivatives of f_3, given by (13), with respect to z, w, y, respectively. Hence the resultant of the final forms is the discriminant Δ of f_3. Let J denote the Jacobian of these derivatives, and j denote the Jacobian of our forms having the resultant r. By a general theorem, j equals the product of J by the Jacobian g^2 of w, z, y with respect to δ, ϵ, τ. Now J is independent of g^2. Hence the exponent of $M = g^2$ in any coefficient of j exceeds by unity the exponent of τ in the term. This is therefore true also of the derivatives j_δ, j_ϵ, while in j_τ the exponent of M in any term exceeds by 2 the exponent of τ. We remove the factors M, M, M^2 from the last three functions. We now have six quadratic forms in δ, ϵ, τ, such that M does not occur as a factor of a coefficient of $\delta^2, \delta\epsilon, \epsilon^2$, while M (but not M^2) is a factor of the coefficients of $\delta\tau$ and $\epsilon\tau$, and M^2 (but not M^3) is a factor of the coefficient of τ^2. Their determinant thus has the factor M^4, but not M^5. Hence $r = M^4 \cdot M^4\Delta$. But $R = M^4r$. Hence $R = M^{12}\Delta$. Thus g^2 is a root of an equation of degree 12 whose leading coefficient is the discriminant of f_3.

* Salmon, Algebra, § 90.

The constant term of this equation is not identically zero. It is in fact not zero when $J = H = 0$, $LG \neq 0$. For, then equations (15)–(17) become for $g = 0$

$$3G(\epsilon^2 - \delta L) = 0, \qquad 3L(\delta^2 - \epsilon G) = 0, \qquad 3LG(-7\delta\epsilon + \tfrac{1}{4}LG) = 0.$$

By the first two, $\delta\epsilon = 0$ or LG. Hence the resultant is not zero.

The coefficients in (15)–(17) are independent of g if and only if A, B, C, D, E, F are all zero. If they are zero the resultant is free of g and, as just shown, is not zero if $J = H = 0$, $LG \neq 0$.

THEOREM. *The problem to express $x^2y + f_3$ rationally in determinantal form, where f_3 is a cubic form in y, z, w, depends completely upon the determination of a rational square which satisfies an equation of degree* 12 *whose leading coefficient is the discriminant of f_3 and whose constant term is not zero identically. The equation reduces to its constant term if and only if f_3 lacks y. In particular, if $LG \neq 0$, $x^2y + Gz^3 + Lw^3$ is not expressible in determinantal form under our present assumptions* (cf. § 10).

The equation for g involves only even powers since this is true of (15)–(17). A more fundamental reason is given in § 14, where there occurs an example in which the equation for g^2 is a cubic, the discriminant of f_3 being zero.

To give another example, take $l_4 \equiv w$, $l_2 \equiv -y$, $l_1 \equiv 0$, $g = 1$. Then

$$\begin{vmatrix} x & -y & w \\ w & x & -z \\ z & w & y \end{vmatrix} = x^2y + w^3 + wy^2 + yz^2,$$

with no rational singular point. Since $C = D = L = 1$ and the remaining coefficients of f_3 are zero, (15)–(17) become

$$2g^2\epsilon = 0, \qquad 3\delta^2 = g^2 - g^4, \qquad g^2\epsilon^2 + (2g^4 - 8g^2)\delta = 0.$$

We desire that $g \neq 0$. The only real solution is $\delta = \epsilon = 0$, $g^2 = 1$. Hence the only rational representations as a determinant of type (11) are the above determinant and that obtained by changing the signs of the elements other than x in the first two rows.

6. We now discard the assumption that $a = b = 0$, but assume that $g \neq 0$, $\Delta \equiv g^2 + ab \neq 0$. It will prove convenient to introduce $d_1 = \Delta d$, $e_1 = \Delta e$ in place of d and e. The last four equations (14) give

$$g\Delta p = \Delta L - be_1, \qquad g\Delta j = -\Delta G - ad_1,$$

$$g\Delta k = 2gd_1 + ae_1 + \delta, \qquad g\Delta n = bd_1 - 2ge_1 + \epsilon,$$

where

$$\delta = -g^2H + g(bG - aJ) - a^2L, \qquad \epsilon = g^2J + g(aL - bH) + b^2G.$$

In (D), (E), (F), the determinant of the coefficients of h, m, c is $2g\Delta$. Hence

$$
\begin{aligned}
2g\Delta h &= \alpha N + \beta ag - \gamma a^2, & \alpha &\equiv -D - d^2 - jn, & N &\equiv ab + 2g^2,\\
2g\Delta m &= -\alpha b^2 + \beta bg + \gamma N, & \beta &\equiv -E - 2de - jp - kn,\\
2g\Delta c &= \alpha bg - \beta g^2 + \gamma ag, & \gamma &\equiv F + e^2 + kp.
\end{aligned}
$$

Replacing p, j, k, n by their values, we get

$$
\begin{aligned}
g^2\Delta^2\alpha &= d_1^2(ab - g^2) - 2d_1e_1ag + d_1(\Delta bG + a\epsilon) - 2e_1\Delta Gg + \Delta G\epsilon - \Delta^2Dg^2,\\
g^2\Delta^2\beta &= -2d_1^2bg + 2d_1e_1(g^2 - ab) + 2e_1^2ag + d_1(\Delta La - b\delta - 2g\epsilon)\\
&\qquad + e_1(2g\delta - a\epsilon - \Delta Gb) - \delta\epsilon - \Delta^2Eg^2 + \Delta^2LG,\\
g^2\Delta^2\gamma &= e_1^2(g^2 - ab) - 2d_1e_1bg + 2d_1\Delta Lg + e_1(\Delta La - b\delta) + \Delta L\delta + \Delta^2Fg^2.
\end{aligned}
$$

Mulitplying equation (B) by $2g^4\Delta^4$ and inserting the preceding values of h, m, c, n, j, we obtain

$$
\begin{aligned}
(18)\quad & 2d_1^3\Delta ab^2 - 2d_1^2e_1\Delta abg + 2d_1e_1^2\Delta a^2b - 2e_1^3\Delta a^2g\\
&+ d_1^2\{\Delta(5Gb^2g^2 + 3Gab^3 - 2Lag^3 - 4La^2bg + 3ab\epsilon) + 2g^3(b\delta + g\epsilon)\}\\
&+ 2d_1e_1\{(\Delta La^2 - \delta N)(g^2 - ab) - 2\Delta GbgN - 2a^2bg\epsilon\}\\
&+ e_1^2\{2\Delta La^3g + (\Delta GN + \epsilon a^2)(ab + 3g^2) - 2\delta gaN\}\\
&+ d_1\{-3\Delta^2LGgN - \Delta La^2(g\epsilon + 2b\delta) + 2\Delta^2Eg^5 - 2\Delta^2Fa^2bg^2\\
&\quad - 2\Delta^2Dbg^2N + \Delta^2G^2b^3 + 4\Delta Gb\epsilon N + \Delta Gb^2g\delta + a^2b\epsilon^2\\
&\qquad + (2g^3 - abg)\delta\epsilon\}\\
&+ e_1\{(a\delta\epsilon - \Delta^2LGa)(ab + 4g^2) + L\Delta a^2(2g\delta - a\epsilon) + 2F\Delta^2g^3a^2\\
&\quad + 2\Delta^2Eag^4 + 2\Delta^2Dg^3N - \Delta^2G^2b^2g - 4\Delta Gg\epsilon N + \Delta Gab^2\delta - a^2g\epsilon^2\}\\
&+ 2\Delta^4g^4B + (\Delta^2Gb^2 + \Delta\epsilon N)(G\epsilon - \Delta Dg^2)\\
&\quad + (ag\epsilon - \Delta Gbg)(\Delta^2LG - \Delta^2Eg^2 - \delta\epsilon)\\
&\qquad - (\Delta^2GN + \Delta a^2\epsilon)(L\delta + \Delta Fg^2) = 0.
\end{aligned}
$$

Similarly, from (C) we obtain an equation

$$(19)\qquad -2d_1^3\Delta b^2g - 2d_1^2e_1\Delta ab^2 - 2d_1e_1^2\Delta abg - 2e_1^3\Delta a^2b + \cdots,$$

which may be deduced from (18) by the substitution*

$$(20)\qquad (BC)(DF)(GL)(HJ)(ab)(d_1, -e_1)(g, -g),$$

which leaves A, E, Δ, N, β unaltered and induces

* $(d, -e)$ replaces d by $-e$ and e by $-d$.

$$(\delta, -\epsilon)(\alpha, -\gamma)(c, -c)(hm)(kn)(jp).$$

These substitutions and (zw) interchange l_2 with l_4, l_3 with l_6, and change the sign of l_1 and hence interchange the first two rows and first two columns of the initial determinant (11).

There remains the first equation (A) in (14). Its product by $4g^6\Delta^5$ may be written in the form

$$4g^6\Delta^5 A - g^4\Delta^4(\alpha b - \gamma a)^2 + g^6\Delta^4(\beta^2 + 4\alpha\gamma) = 0. \tag{21}$$

Its terms of the highest degree in d_1, e_1 are $(3g^2 - ab)\lambda$, where

$$\lambda = \Delta(d_1^4 b^2 + 2d_1^2 e_1^2 ab + e_1^4 a^2).$$

The terms of the fourth degree in $bd_1(18) - ae_1(19)$ are $2ab\lambda$; those in $ge_1(18) + gd_1(19)$ are $-2g^2\lambda$. Hence by adding multiples of (18) and (19) to (21), we may cancel the terms of the fourth degree. Hence we have three cubic functions of d_1, e_1, whose true resultant R is known to be expressible as a determinant.

We proceed to find the terms of R of maximum degree in g. From the aggregate of terms in (18) which multiply each $d_1^r e_1^s$ we omit all terms not of the highest degree in g, remove the common factor 2 and obtain

$$d_1^3 g^2 ab^2 - d_1^2 e_1 g^3 ab + d_1 e_1^2 g^2 a^2 b - e_1^3 g^3 a^2 + d_1^2 g^6 J + 2d_1 e_1 g^6 H + 3e_1^2 g^6 G + d_1 g^9 E + 2e_1 g^9 D + g^{12} B. \tag{22}$$

Applying to this the substitution (20), we obtain

$$-d_1^3 g^3 b^2 - d_1^2 e_1 g^2 ab^2 - d_1 e_1^2 g^3 ab - e_1^3 g^2 a^2 b + 3d_1^2 g^6 L + 2d_1 e_1 g^6 J + e_1^2 g^6 H + 2d_1 g^9 F + e_1 g^9 E + g^{12} C. \tag{23}$$

The resultant of these two functions and a third function T of d_1, e_1 is the same as the resultant of their sum and difference and T. Hence we may omit the terms involving only g^2. Remove the factors g^3, make the functions homogeneous in d_1, e_1, τ and set $g^3\tau = \phi$; we get the forms free of g:

$$-d_1^2 e_1 ab - e_1^3 a^2 + d_1^2\phi J + 2d_1 e_1 \phi H + 3e_1^2 \phi G + d_1\phi^2 E + 2e_1\phi^2 D + \phi^3 B, \tag{24}$$

$$-d_1^3 b^2 - d_1 e_1^2 ab + 3d_1^2\phi L + 2d_1 e_1\phi J + e_1^2\phi H + 2d_1\phi^2 F + e_1\phi^2 E + \phi^3 C. \tag{25}$$

In (21) the terms of maximum degrees in g are

$$3d_1^4 g^4 b^2 + 6d_1^2 e_1^2 g^4 ab + 3e_1^4 g^4 a^2 - 8d_1^3 g^7 L - 8d_1^2 e_1 g^7 J - 8d_1 e_1^2 g^7 H - 8e_1^3 g^7 G - 4d_1^2 g^{10} F - 4d_1 e_1 g^{10} E - 4e_1^2 g^{10} D + d_1 g^{11}(4JE - 8LD) + e_1 g^{11}(4HE - 8GF) + 4g^{16} A. \tag{26}$$

The terms in g^{11} near the end do not contribute to the part of the resultant

of highest degree in g, since by omitting them (or replacing them by g^{13} with the coefficient zero), we shall obtain a resultant not identically zero. Also divide by $4g^4$, make homogeneous in d_1, e_1, τ, and set $g^3\tau = \phi$ as before; we get the form free of g:

$$(27)\quad \tfrac{3}{4}d_1^4b^2 + \tfrac{3}{2}d_1^2e_1^2ab + \tfrac{3}{4}e_1^4a^2 - 2d_1^3\phi L - 2d_1^2e_1\phi J - 2d_1e_1^2\phi H - 2e_1^3\phi G \\ - d_1^2\phi^2F - d_1e_1\phi^2E - e_1^2\phi^2D + \phi^4A.$$

Now (24) and (25) are the partial derivatives with respect to e_1 and d_1 of

$$(28)\quad A\phi^4 + Be_1\phi^3 + Cd_1\phi^3 + De_1^2\phi^2 + Ed_1e_1\phi^2 + Fd_1^2\phi^2 + Ge_1^3\phi + Hd_1e_1^2\phi \\ + Jd_1^2e_1\phi + Ld_1^3\phi - \tfrac{1}{4}d_1^4b^2 - \tfrac{1}{2}d_1^2e_1^2ab - \tfrac{1}{4}e_1^4a^2 \\ \equiv \phi f_3(\phi, e_1, d_1) - \tfrac{1}{4}(d_1^2b + e_1^2a)^2,$$

where $f_3(y, z, w)$ is given by (13). To (27) add the product of (24) by e_1 and the product of (25) by d_1; we get (28). Hence the resultant of (24), (25), (27) equals the resultant of (28) and its partial derivatives, i.e., the discriminant of (28). This discriminant is not zero for all values of $A, \cdots, L, a, b$. In fact, the general quartic curve becomes $zK - x^2y^2 = 0$ when referred to a triangle of reference whose side $z = 0$ is a bitangent and whose sides $x = 0$ and $y = 0$ are any lines through its two points of contact. Evidently (28) is of this form if x, y are the factors of $\tfrac{1}{2}(d_1^2b + e_1^2a)$.

The quartic curve (28) is a plane section* of the tangent cone to the cubic surface (1), viz.,

$$(1')\qquad x^2\phi + x(ae_1^2 + bd_1^2) + f_3(\phi, e_1, d_1) = 0$$

whose vertex is $\phi = d_1 = e_1 = 0, x = 1$. A point on (1′) will be a singular point if the partial derivatives with respect to x, e_1, d_1 vanish:

$$2x\phi + ae_1^2 + bd_1^2, \qquad 2axe_1 + \frac{\partial f_3}{\partial e_1}, \qquad 2xbd_1 + \frac{\partial f_3}{\partial d_1}.$$

Let $\phi \neq 0$ and substitute the value of x for which the first function vanishes into the others and (1′), and multiply each result by ϕ; we get

$$(29)\quad - ae_1(ae_1^2 + bd_1^2) + \phi\frac{\partial f_3}{\partial e_1}, \qquad - bd_1(ae_1^2 + bd_1^2) + \phi\frac{\partial f_3}{\partial d_1}, \\ - \tfrac{1}{4}(ae_1^2 + bd_1^2)^2 + \phi f_3,$$

which equal functions (24), (25), (28), respectively. Since a general cubic surface is reducible to (1′) by § 2, and has no singular point, we again conclude that the resultant of (24), (25), (27) is not identically zero.

* Miller, Blichfeldt, Dickson, Finite Groups, 1916, p. 352.

Multiply the first function (29) by e_1 and the second by d_1, add and apply Euler's theorem $\Sigma e_1 \partial f_3/\partial e_1 = 3f_3$; we get

$$e_1(24) + d_1(25) = -(ae_1^2 + bd_1^2)^2 + \phi\left(3f_3 - \phi\frac{\partial f_3}{\partial \phi}\right).$$

Subtract this from 4 times $e_1(24) + d_1(25) + (27) = (28)$, given above. Hence

$$3e_1(24) + 3d_1(25) + 4(27) = \phi\frac{\partial(\phi f_3)}{\partial \phi}.$$

Hence we may replace (27) by the cubic function $\partial(\phi f_3)/\partial\phi$. This may therefore be obtained by deleting the factor g^4 from

$$3e_1 g(22) + 3d_1 g(23) + (26),$$

after the three functions are abridged as above by omission of terms in g^2 and g^{11}. We also deleted the factor g^3 from each of the abridged cubics (22), (23). But the resultant is of the ninth degree in the coefficients of each form, and hence equals the product of $(g^3)^9(g^3)^9(g^4)^9 = g^{90}$ by the resultant r of the three cubic forms homogeneous in d_1, e_1, τ. Their Jacobian j equals the product of the Jabocian J of the equivalent forms in d_1, e_1, $\phi = g^3\tau$, by the Jacobian g^3 of d_1, e_1, ϕ with respect to d_1, e_1, τ. But J is independent of g. Hence the exponent of $M = g^3$ in any coefficient in j exceeds by unity the exponent of τ in the term. The same is therefore true of the derivatives j_{d_1} and j_{e_1}, while in j_ϕ the exponent of M exceeds by 2 the exponent of τ in any term. We remove the factors M, M, M^2 and obtain three quintic functions in which the exponent of M in any coefficient equals the exponent of τ in the term. The same is true of each of our three cubic forms. If we multiply each of them by d_1^2, d_1e_1, e_1^2, $d_1\tau$, $e_1\tau$, τ^2, we obtain 18 quintic forms. The determinant (of order 21) of the coefficients in these and the former three quintics is known* to be the resultant ρ of the three cubics. In each of the three equations obtained by use of the multiplier τ^2, the exponent of M in any term is 2 less than the exponent of τ; we multiply these equations by M^2. Similarly we multiply by M each of the six equations obtained by use of the multipliers $d_1\tau$, $e_1\tau$. Now in all 21 equations the exponent of M in any term equals the exponent of τ. Hence in the new determinant the elements of any column contain the same power of M as a factor, viz., the exponent of τ in the corresponding term $d_1^l e_1^m \tau^n$, $l + m + n = 5$. Hence the total exponent of the power of M which divides the determinant is $5 + 4(2) + 3(3) + 2(4) + 1(5) = 35$. But our new determinant equals $\rho(M^2)^3M^6$. Hence ρ is the product of $M^{35}/M^{12} = M^{23}$ by a constant $\lambda \neq 0$. Thus, accounting for M^4 removed

* Salmon, Algebra, § 90.

above, $r = M^{27}\lambda$. Hence the resultant of our initial equations is of degree $90 + 3\cdot 27 = 171$ in g.

THEOREM. *When the quadratic form f_2 in z, w does not have rational factors, the problem to express $x^2y + xf_2 + f_3$ rationally in determinantal form depends completely upon the solution of an equation of degree* 171 *whose leading coefficient is not zero if the surface has no singular point.*

Perhaps this equation is reducible since its degree is high in comparison with the degree of the equation upon which depends the determination of the 27 ruled lines on the surface.

Material simplifications arise if $G = H = J = L = 0$, whence y is a factor of f_3, and $y = 0$ cuts the surface in ruled lines. Since $\delta = \epsilon = 0$, (18) and (19) have no quadratic terms. Dividing $bd_1(18) - ae_1(19)$ and $e_1(18) + d_1(19)$ by 2, we get

$$(30)\quad \Delta ab(d_1^2b + e_1^2a)^2 + d_1^2\Delta^2g^2(Eg^3b - Fa^2b^2 - Db^2N) - e_1^2\Delta^2g^2(Eg^3a + Fa^2N + Da^2b^2) + 2d_1e_1\Delta^2g^4(Eab - Fag + Dbg) + \Delta^4g^4(d_1Bb - e_1Ca) = 0,$$

$$(31)\quad -\Delta g(d_1^2b + e_1^2a)^2 + d_1^2\Delta^2g^3(-Egb + FN + Db^2) + e_1^2\Delta^2g^3(Ega + Fa^2 + DN) + 2d_1e_1\Delta^2g^4(Eg + Fa - Db) + \Delta^4g^4(d_1C + e_1B) = 0.$$

Dividing $g(30) + ab(31)$ by Δ^2g^4, we get

$$(32)\quad (bd_1^2 - ae_1^2)[E(g^2 - ab) + 2Fga - 2Dbg] + 2d_1e_1[2Eabg + (g^2 - ab)(Db - Fa)] + \Delta^2bd_1(gB + Ca) + \Delta^2ae_1(Bb - Cg) = 0.$$

We obtain a second quadratic from $[(21) + (30) + 3g(31)]/\Delta^2g^2$:

$$(33)\quad d_1^2[2Ebg^3+F\rho+D(ab^3-g^2b^2)]+e_1^2[-2Eag^3+F(a^3b-g^2a^2)+D\rho] +2d_1e_1g^2[E(g^2+3ab)-2Fag+2Dbg]+d_1\Delta^2g^2(3Cg+Bb) +e_1\Delta^2g^2(3Bg-Ca)+\Delta^2g^4(E^2-4FD+4\Delta A)-\Delta^2g^2(Db+Fa)^2=0,$$

where $\rho = a^2b^2 + 5abg^2 + 2g^4$. Here the coefficients of d_1^2 and e_1^2 will be proportional to b and $-a$, as in (32), if and only if $aF + bD = 0$, and then the same fact holds in (30) and (31). If also $B = C = 0$, no first degree terms in d_1, e_1 occur in our equations. Then (32) and (33) give

$$(bd_1^2-ae_1^2)Q=4\Delta[Eabg-(g^2-ab)Fa]Sg^2,\quad d_1e_1Q=\Delta[-E(g^2-ab)-4Fag]Sg^2,$$

$$Q = (g^2 - 3ab)t,\qquad t = E^2 + 4F^2a/b,\qquad S = 2\Delta A + \tfrac{1}{2}t.$$

From the squares of these, we get (if* $t \neq 0$)

$$(bd_1^2 + ae_1^2)^2(g^2 - 3ab)Q = 4ab\Delta^4 g^4 S^2.$$

We obtain a second expression for the left member by use of $3(30) + g(31)$:

$$-(bd_1^2 + ae_1^2)^2(g^2 - 3ab)\Delta + (bd_1^2 - ae_1^2)\Delta^2 g^4\left[2Eg + \frac{F}{b}(6ab + 2g^2)\right]$$
$$+ d_1e_1\Delta^2 g^3[2E(g^3 + 3abg) - 8Fag^2] = 0.$$

Multiplying this by Q, inserting the preceding values, and cancelling the common factor $-2\Delta^4 g^4$, we get either $S = 0$ or

$$g^4(4abA + t) + 2abg^2(4abA - t) + a^2b^2(4abA + t) = 0,$$

whence

$$\{g^2(4abA + t) + ab(4abA - t)\}^2 = -16a^3b^3At.$$

Hence the number of rational values of g^2 is 3 or 1 according as $-abAt$ is or is not a rational square. If also $A = D = -1$, $E = 2$, $F = 1$, $a = b = 1$, then $g = \pm 1, \pm 1 + \sqrt{2}, \pm 1 - \sqrt{2}$, and the surface has no singular point.

7. In §§ 7–11 we shall treat the special cases which were excluded in §§ 5–6. In each case the determinantal surface has a known ruled line whose equations are so simple that its occurrence on the given surface can be detected by inspection. When the given surface has the line as a ruling, its representation as a determinant is a much simpler problem than that treated in §§ 5–6. Accordingly one should first ascertain whether or not the given surface falls under one of these special cases.

Consider the case $g^2 + ab = 0$ which was excluded in § 6. If $a = 0$, then $b = 0$ (§ 5) and $g = 0$, $l_3 = l_6 = 0$, so that (11) is the product of y by its minor, a case treated at the end of § 11. Hence let $a \neq 0$. We first treat the case $b \neq 0$. Then

$$az^2 + bw^2 = bZW, \quad Z = w + \frac{a}{g}z, \quad W = w - \frac{a}{g}z, \quad l_3 = gW, \quad l_6 = -bW.$$

Introduce Z and W as new variables in place of z and w and let the new f_3 be given the same notation (13). We have therefore to consider a form (1) in which $f_2 = bzw$. Interchanging z and w if necessary, we may assume that, in (2), l_7 contains z. Proceeding exactly as in § 5, we may set $l_8 = w$, $l_7 = z$. By (3), $l_6 = tz$, $l_3 = (1 - t)w$. But l_3 and l_6 must be dependent and hence one of them zero. Applying substitution S, we may take $l_6 \equiv 0$.

* If $t = 0$, either $Sg^2 = 0$ or $E = F = 0$, since the determinant of the coefficients of E and F equals $-\Delta^2$. For $E = F = B = C = 0$, $D = 0$, (33) gives $A = 0$, and f has the factor x.

Thus we have (11) with $l_6 \equiv 0$, $l_3 = w$. It remains to identify the final determinant in (11) with f_3. Set $l_1 = cy + \lambda_1$, $l_4 = my + \lambda_4$. By the terms free of y,

$$w(w\lambda_4 + z\lambda_1) \equiv Gz^3 + Hz^2w + Jzw^2 + Lw^3,$$

whence $G = 0, \lambda_4 = Lw + nz, \lambda_1 = Hz + (J - n)w$. From the remaining terms we remove the factor y and have

$$- l_1^2 - l_2 l_4 + w(wm + zc) \equiv Ay^2 + Byz + Cyw + Dz^2 + Ezw + Fw^2.$$

The resulting conditions $(A), \cdots, (F)$ are

$$- c^2 - mh = A, \quad - 2cH - mj - nh = B, \quad - 2c(J - n) - mk - Lh = C,$$

$$- H^2 - nj = D, \quad c - 2H(J - n) - nk - Lj = E, \quad m - Lk - (J - n)^2 = F.$$

First, let $n \neq 0$. From $(D), (E), (F), (B)$,

$$j = \frac{-N}{n}, \quad k = \frac{P + cn}{n^2}, \quad m = \frac{Lc}{n} + \frac{R}{n^2}, \quad h = \frac{-B - 2cH}{n} + \frac{N}{n^2}\left(\frac{Lc}{n} + \frac{R}{n^2}\right),$$

where

$$N = D + H^2, \qquad P = LN - n(E + 2HJ) + 2Hn^2,$$

$$R = n^2F + (J - n)^2 n^2 + LP.$$

To clear the denominators of n multiply (A) by n^6, (C) by n^4, and use the abbreviations

$$N = D + H^2, \qquad S = E + 2HJ, \qquad T = F + J^2 + 2LH,$$

$$R = n^4 - 2n^3J + n^2T - nLS + L^2N.$$

Hence

$$(34) \quad c^2(n^6 - 2n^4LH + n^2L^2N) + c[2R(nLN - n^3H) - n^4LB] + R^2N - Rn^3B + n^6A = 0,$$

$$(35) \quad c^2n^2L + c(- n^5 + n^3T - 2n^2LS + 3nL^2N) + R(2n^2H - nS + 2LN) - n^3LB + n^4C = 0.$$

Retaining in each coefficient only the highest power of n, we get

$$c^2n^6 - 2cn^7H + n^8N = 0, \qquad c^2n^2L - cn^5 + 2n^6H = 0,$$

whose resultant is $n^{24}N$. Hence n satisfies an equation of degree 24.

Second, let $n = 0$, $L \neq 0$. This case occurs only if $G = 0$, $D = - H^2$. Use the abbreviations $S = E + 2HJ$, $\beta = F + J^2$. Then $(E), (F), (C)$ give

$$Lj = c - S, \qquad m = \beta + Lk, \qquad Lh = -C - 2cJ - k(\beta + Lk).$$

By (B), $c = \lambda/\mu$, $\lambda = -BL + S(\beta + Lk)$, $\mu = 2HL + \beta + Lk$. Then (A) becomes a quintic in k, the coefficient of k^5 being $-L^4$.

Third, if $n = L = 0$, (F) and (E) give m and c. If $m \neq 0$, (C), (B), (A) give k, j, h, so that there is an unique rational solution.

8. There remains the case $b = g = 0$, $a \neq 0$. Multiplying x by $1/a$, and y by a^2, we may set $a = 1$. Hence we examine conditions (14) when $b = g = 0$, $a = 1$.

First, let $J \neq 0$. Replacing w by $w - zH/(2J)$, we have $H = 0$. Then equations (14), other than the first two, give

$$L = 0, \qquad p = -J, \qquad n = -e, \qquad d = -G, \qquad k = (e^2 + F)/J,$$

$$c = je - G^2 - D, \qquad m = Jj + 2Ge + ke - E, \qquad h = (C + 2ce + km)/J.$$

We retain the abbreviations k and $\rho = 2Ge - E + ke$, whence $m = Jj + \rho$. Equations (B) and (A) of (14) become

$$Jj^2 - Ej + \alpha = 0, \quad \alpha \equiv B + 2G(G^2 + D) - (Ce + 2ce^2)/J - ke\rho/J,$$

$$(2e^2 + F)j^2 + \beta j + \gamma = 0, \quad \beta \equiv 2\rho k + C + 2ce - 2e(G^2 + D),$$

$$\gamma \equiv A + (G^2 + D)^2 + (C + 2ce)\rho/J + k\rho^2/J.$$

If in each coefficient of these two quadratics in j we retain only the term of maximum degree in e, we find that their resultant becomes $5e^{16}/J^6$. Hence e is a root of an equation of degree 16.

Second, let $J = 0$. Then equations (14), other than the first three, give

$$L = 0, \qquad p = 0, \qquad n = -e - H, \qquad d = -G, \qquad e^2 = -F,$$

$$c = j(e + H) - G^2 - D, \qquad m = k(e + H) + 2Ge - E.$$

The possibility of a (unique) solution with $e = -H$ is easily decided since equations (A), (B), (C) determine uniquely h, j, k if $m \neq 0$. When $e \neq -H$, $e \neq 0$, we express the unknowns in terms of k and e, retaining the abbreviation m. By (C), $c = (-C - km)/(2e)$. Equating this to the above value of c, we get j. Then (B) gives h. Hence (A) becomes

$$4Ae^2(e + H)^2 + (C + km)^2(e + H)^2 + 4Bme^2(e + H)$$
$$+ 4(C + km)Gme(e + H) + 2m^2e\{2(G^2 + D)e - C - km\} = 0,$$

which is a quartic in k, the coefficient of k^4 being $(H + e)^3(H - e)$.

9. Consider the case, excluded in § 5, in which l_3, l_6, l_7, l_8 are all free of z and hence are multiples of w. Postponing to the end of § 11 the case in which (11) is the product of y by its minor, we may assume, in view of the

substitutions S and T of § 3, that l_7 is the product of w by a constant, not zero, which may be removed as a factor from the last row and multiplied into the last column of (2). Thus $l_7 = w$. We may take $l_8 = 0$ by (4). Then $-l_3 w = az^2 + bw^2$ by (3). Thus $a = 0$ and hence $b = 0$ by § 5. Then $l_3 \equiv 0$. Then $l_6 \not\equiv 0$ and by removing its coefficient from the second row and multiplying it into the second column, we may set $l_6 = w$. Hence

$$f = \begin{vmatrix} x + l_1 & l_2 & 0 \\ l_4 & x - l_1 & w \\ w & 0 & y \end{vmatrix} = x^2 y - y(l_1^2 + l_2 l_4) + w^2 l_2.$$

For $y = 0$, f reduces to $w^2\lambda$, if $l_2 = hy + \lambda$. Hence in

$$f_3 = -yQ + K, \qquad Q = Ay^2 + 2Byz + 2Cyw + Dz^2 + 2Ezw + Fw^2,$$

the cubic function K of z, w must have the factor w^2, the quotient being $\lambda = jz + kw$. Further, Q must be of the form $l_1^2 + l_2 l_4 - kw^2$. We shall examine the last question independently of our main problem. If $j \neq 0$, we introduce the known function λ as a new variable z; let Q become $Q' = A'y^2 + \cdots$. By (5) we may subtract a multiple of $l_2 = hy + z$ from l_1 and assume that $l_1 = cy + ew$ lacks z. When Q' is divided by l_2 to give a remainder free of z, the quotient gives l_4 and the remainder must be $l_1^2 - hw^2$. Hence the latter must equal the value of Q' for $z = -hy$, the conditions for which are

$$c^2 = A' - 2B'h + D'h^2, \qquad ce = C' - E'h, \qquad e^2 = h + F'.$$

Hence the cubic

$$(A' - 2B'h + D'h^2)(h + F') = (C' - E'h)^2$$

must have a rational root h such that $h + F'$ is a rational square e^2.

Next, let $j = 0$. Then $k \neq 0$ since we assume that y is not a factor of f. We may take $l_1 = cy + dz$ free of w by (15). When Q and $l_1^2 - hw^2$ are divided by l_2, the remainders must be equal and a comparison of the quotients determines l_4. Thus the function obtained from Q by replacing w by $-yh/k$ must be identical with $(cy + dz)^2 - y^2h^3/k^2$, whence

$$d^2 = D, \qquad cd = B - \frac{Eh}{k}, \qquad c^2 = A - 2\frac{Ch}{k} + F\frac{h^2}{k^2} + \frac{h^3}{k^2}.$$

10. Consider the case $v = 0$, excluded in § 5, in which $l_8 = 0$, $l_7 = z + uw$. By (3), $l_3 = -a(z - uw)$, $b = -au^2$.

If $a = b = 0$, determinant (2) now equals

$$x^2 y - y(l_1^2 + l_2 l_4) + l_2 l_6 l_7. \tag{36}$$

This shall equal $f \equiv x^2y - yQ + K$, for Q and K as in § 9. Thus $K = \lambda l_6 l_7$. We postpone to § 11 the simple case $K \equiv 0$. One of the factors of K determines $l_7 = z + uw$. The other two factors determine λ and l_6 up to constant factors; but our determinant (36) is not altered when l_6 and l_4 are multiplied by $t \neq 0$ and $l_2 = hy + \lambda$ is multiplied by $1/t$. Hence we may regard λ, l_6 and l_7 as fully determined. We introduce λ as a new variable z and proceed as in § 9.

Next, let $a \neq 0$. In view of (5) we may take $l_6 = rw$. If $r \neq 0$, we apply the substitution $(l_2l_4)(l_3l_7)(l_6l_8)$, which corresponds to the interchange of rows and columns, and have the case $v \neq 0$ treated in § 5. If $r = 0$, the determinant is identical with (1) if

$$-y(l_1^2 + l_2l_4) - al_1(z + uw)(z - uw) \equiv f_3,$$

a condition which is exactly of the type last discussed.

11. Consider factorable cubic forms yQ, hitherto excluded. If Q itself has the factor y, yQ equals a determinant whose elements outside the diagonal are all zero. In the contrary case, we may take $Q = cx^2 + \cdots$, where $c \neq 0$, after applying a linear transformation on x, z, w with rational coefficients. Replacing cy by a new y, we have yQ, the coefficient of x^2 in Q being unity. Making a suitable addition to x, we obtain $Q = x^2 + q$, where q is a quadratic form in y, z, w. Thus yQ is of the form (1) with $f_2 \equiv 0, f_3 \equiv yq$.

First, consider equations (14) when $a = b = G = H = J = L = 0$, $g \neq 0$. The last seven give at once

$$p = j = 0,\ k = 2d,\ n = -2e,\ gm = F + e^2,\ gh = -D - d^2,\ 2gc = E - 2de.$$

Multiplying the first equation (14) by $-g^2$ and the next two by g, we get

$$dE + 2eD + gB = 0,\ 2dF + eE + gC = 0,\ d^2F + e^2D + deE = Ag^2 + \tfrac{1}{4}\Delta,$$

where $\Delta = E^2 - 4DF$. Multiply them by e, d, -2, respectively, and add. We get

$$\begin{aligned} 2g^2A + geB + gdC &= -\tfrac{1}{2}\Delta, \\ g^2B + 2geD + gdE &= 0, \\ g^2C + geE + 2gdF &= 0, \end{aligned}$$

the last two being our first two equations multiplied by g. The determinant of the coefficients of g^2, ge, gd is 8 times the determinant δ of

(37) $$q = Ay^2 + Byz + Cyw + Dz^2 + Ezw + Fw^2.$$

Hence $8\delta g^2 = -\tfrac{1}{2}\Delta(-\Delta)$. Thus if $\delta \neq 0$, $\Delta \neq 0$, the equations have solutions g, e, d with $g \neq 0$, and g will be rational if δ is a square. If $\Delta = 0$,

the equations have no solutions with $g \neq 0$ if any two-rowed minor is not zero. But if those minors are all zero, there is a solution with $g \neq 0$ except when $q = Ay^2$, $A \neq 0$. Hence the cubic $y(x^2 + q)$ has a rational determinantal representation of the type in § 5 only when the determinant of q is a rational square $\neq 0$ and its minor $E^2 - 4DF$ is not zero, or in the trivial case when q is a perfect square. If we make the substitution $z = Z + eY$, $w = W + dY$, $y = gY$, we see that q becomes $DZ^2 + EZW + FW^2 - \frac{1}{4}\Delta Y^2$ and that our determinant of type (11) reduces to the product of g by (38), which is the value of the initial determinant when $d = e = B = C = 0$, $g = 1$.

Milder restrictions are imposed by the method of § 10 for $a = b = 0$, $K \equiv 0$. Thus $\lambda l_6 l_7 \equiv 0$. If $l_6 \equiv 0$, the determinant equals the product of y by its minor, a case treated below. Hence $\lambda \equiv 0$, $l_2 = hy$. The case $h = 0$ is of the type just postponed. Writing hl_4 as a new l_4 in (36), we may take $h = 1$. In view of (5) we may assume that l_1 is free of y. Our cubic is of the form (36) if $q \equiv -l_1^2 - yl_4 + l_6 l_7$. Hence, by (37), $-l_4 = Ay + Bz + Cw$. It remains to choose

$$l_1 = dz + ew, \qquad l_6 = rz + sw, \qquad l_7 = vz + uw$$

so that $Dz^2 + Ezw + Fw^2 \equiv -l_1^2 + l_6 l_7$. The conditions are

$$vr = d^2 + D, \qquad ur + vs = 2de + E, \qquad us = e^2 + F.$$

These are linear equations in $r, s, -1$, the determinant of whose coefficients is

$$u^2(d^2 + D) - uv(2de + E) + v^2(e^2 + F) = 0,$$

$$-u^2 D + uvE - v^2 F = (ud - ve)^2.$$

The problem is solvable if we can choose rational numbers u, v, not both zero, such that the left member* is the square ρ^2 of a rational number. Then rational values of d, e may be chosen so that $ud - ve = \rho$. Hence $y(x^2 + q)$ has a rational determinantal representation of the present type if and only if $x^2 + q$ vanishes at a rational point having $y = 0$.

If our cubic is of the form in § 9, we must have $\lambda \equiv 0$, and

$$q = -l_1^2 - hyl_4 + hw^2,$$

where we may assume that $h \neq 0$ and that l_1 is free of y (as in the preceding case). Thus $-hyl_4$ equals the sum of the first three terms of (37) and $-l_1^2 + hw^2$ equals the sum of the last three. The conditions on $l_1 = dz + ew$ are $d^2 = -D$, $2de = -E$, $h = e^2 + F$. These determine d, e, h rationally if $-D$ is a rational square $\neq 0$, and are satisfied if $D = E = 0$. In the

* That this condition is necessary is seen by taking $z = u$, $w = -v$ in the proposed identity, whence $l_7 = 0$.

respective cases, $x^2 + q$ vanishes at a rational point having $y = w = 0$ or $x = y = w = 0$.

It remains to consider the frequently postponed case of the rational representation of a product of a linear form l and a quadratic form Q as a determinant in which the elements of a row or column are $0, 0, l$. Thus Q is to be represented as a two-rowed determinant. An evident necessary condition is that Q vanish at a rational point. Then, as in § 2, Q can be transformed rationally into $xy + q(z, w)$ or $q(y, z, w)$. Any two-rowed determinant equal to the former may be given the form

$$\begin{vmatrix} x + A & B \\ C & y \end{vmatrix},$$

where B and C are free of x and y, whence $A \equiv 0$, $q = -BC$. Similarly, if $q(y, z, w)$ equals a rational determinant, it vanishes for rational values not all zero of y, z, w, and is rationally equivalent to $yz + kw^2$ or to a binary form. In the following summary, the various cases are presented in reverse order.

THEOREM. *A product of a linear form l and quadratic form Q in four variables with rational coefficients can be expressed as a determinant whose elements are linear functions with rational coefficients only in the following cases*: (i) *Q is expressible rationally as a two-rowed determinant if it represents a cone which vanishes at a rational point not the vertex, or if Q vanishes at a rational point P and the tangent plane at P cuts the surface in rational lines.* (ii) *Q and l both vanish at the same rational point.* (iii) *lQ is rationally equivalent to $y(x^2 + q)$, where q is the ternary form* (37) *whose determinant is the square of a rational number $\neq 0$ and its minor $\Delta = E^2 - 4DF$ is not zero; the resulting representation is derived from the special case*

$$(38)\quad Y(x^2 + DZ^2 + EZW + FW^2 - \tfrac{1}{4}\Delta Y^2) = \begin{vmatrix} x + \frac{1}{2}EY & -DY & W \\ FY & x - \frac{1}{2}EY & -Z \\ Z & W & Y \end{vmatrix}$$

by a linear substitution of type $z = Z + eY$, $w = W + \alpha Y$, $y = gY$.

COROLLARY. If A, D, F are rational numbers for which

$$Q = x^2 + AY^2 + DZ^2 + FW^2$$

vanishes at no rational point, YQ is representable rationally as a determinant if and only if ADF is a rational square.

I have found a simple proof of this Corollary independently of the present theory.

12. In view of §§ 5, 9, 10, $x^2y + Gz^3 + LW^3$ is expressible rationally in determinantal form only when $Gz^3 + LW^3$ is a product of three factors with rational coefficients whence $GL = 0$. But if K is any homogeneous cubic, $x^2y - K(x, z, w) = 0$ evidently has the solutions

$$x = \rho A^3, \qquad y = \rho K(A, B, C), \qquad z = \rho A^2 B, \qquad w = \rho A^2 C,$$

and no further rational solutions when $x \neq 0$. For, if A is any rational number $\neq 0$, we may define ρ by $x = \rho A^3$, B by $z = \rho A^2 B$, C by $w = \rho A^2 C$; then y has the value specified.

These solutions do not imply a determinantal representation of the surface since there do not exist three linearly independent linear relations between $x, \cdots, w$ with coefficients linear in A, B, C, provided K does not have the factor x. In fact, all such relations are linear combinations of $Bx - Az = 0$, $Cx - Aw = 0$. But if $K \equiv y^2 z$, for example, we may secure such relations by taking new parameters $A, B, D = A^2/C$; then, for $\rho = \sigma D/A^2$,

$$x = \sigma AD, \qquad y = \sigma B^2, \qquad z = \sigma BD, \qquad w = \sigma A^2,$$

$$Bx - Az = 0, \qquad -Ax + Dw = 0, \qquad -Bz + Dy = 0.$$

13. The question whether the special cases postponed in § 5 are really exceptional is best answered by a different approach to the problem of normalizing our initial determinant. Its matrix is $M = xA + yB + zC + wD$, where, by (2),

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} t_1 & t_2 & 0 \\ t_4 & -t_1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

t_i being the coefficient of y in l_i. To further normalize M, we have available any constant matrices P and Q, whose determinants are not zero, such that in PMQ the coefficient of x is our A, while that of y is a matrix B' of type B with t_i replaced by t'_i. Thus $PAQ = A$, $PBQ = B'$. Set $R = P^{-1}$. Then $AQ = RA$, $BQ = RB'$, which are easily seen to require that

$$R = Q = \begin{bmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & \rho \end{bmatrix}.$$

Thus $PMQ = R^{-1}MR$, so that the normalization of M must arise by transformation by a matrix R of the form just given. Let c_{ij} and d_{ij} denote the elements of C and D in the ith row and jth column. By (2), $c_{33} = d_{33} = 0$, $c_{22} = -c_{11}$, $d_{22} = -d_{11}$.

The first case postponed in § 5 is that in which c_{13}, c_{23}, c_{31}, and c_{32} are all zero. Since $R^{-1}CR$ then has the same four elements zero, this case is truly exceptional. Excluding it, we may take* $c_{31} \neq 0$. By choice of ρ,

* For $\alpha = \delta = 0$, $\beta = \gamma = \rho = 1$, transformation of C by R interchanges c_{13} and c_{23}, c_{31} and c_{32}. We also allow passing to the transposed matrix (with rows and columns interchanged), thus treating one of two similar problems. Note that the transposed matrix cannot be obtained from C by transforming by a matrix of the specal form R.

we may take $c_{31} = 1$. Transforming by R with $\alpha = \delta = \rho = 1$, $\gamma = 0$, $\beta = -c_{32}$, we may now have $c_{32} = 0$ as well as $c_{31} = 1$, $c_{33} = 0$. If also the last row of $R^{-1}CR$ is 1, 0, 0, we must have $\beta = 0$, $\alpha = \rho$. The last row of $R^{-1}DR$ is then

$$d_{31} + \frac{\gamma}{\rho} d_{32}, \qquad \frac{\delta}{\rho} d_{32}, \qquad 0.$$

Hence the special case in which d_{32} (denoted by v in § 5) is zero is truly exceptional. Excluding it, we may choose γ/ρ and δ/ρ so that $d_{31} = 0$, $d_{32} = 1$. Then if R is such that the last row of $R^{-1}DR$ is still 0, 1, 0, we have $\gamma = 0$, $\delta = \rho$, so that R is a similarity-matrix having the diagonal elements equal and having the remaining elements all zero. Thus R transforms every matrix into itself and no further normalization of B, C, D is possible.

14. The difficult part of our problem is to identify the final determinant in (11) with any given ternary form (13) in y, z, w. The matrix of that determinant is $M = yY + zZ + wW$, where

$$Y = \begin{bmatrix} c & h & 0 \\ m & -c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Z = \begin{bmatrix} d & j & -a \\ n & -d & -g \\ 1 & 0 & 0 \end{bmatrix}, \qquad W = \begin{bmatrix} e & k & g \\ p & -e & -b \\ 0 & 1 & 0 \end{bmatrix}.$$

Let Y', Z', W' denote the similar matrices in c', $\cdots$, g', but with the same a and same b. Do there exist matrices P and Q with constant elements of determinants not zero such that $PMQ = M'$? If so, the determinants of M and M' differ only by a constant factor which we may assume is unity. It is more convenient to treat $MQ = RM'$, where $R = P^{-1}$. There are really several questions, depending upon what is assumed to be given.

First, let only the determinant of M be given and require all matrices M, M', $\cdots$ of our special form and investigate their equivalence. Since this is our initial difficult problem with a supplement, we will illustrate the facts by means of an instructive example. Let $|M| = w^3 - z^3 - yzw$, so that $E = G = -1$, $L = 1$, while the remaining coefficients of (13) are zero.* Thus (15)–(17) become

$$-3\epsilon^2 + \delta(3 - g^2) = 0, \quad 3\delta^2 + \epsilon(3 - g^2) = 0, \quad \delta\epsilon(21 - g^2) + \tfrac{3}{4}(1 - g^2)^2 = 0.$$

By the first two, $\delta\epsilon = 0$ or $-(3 - g^2)^2/9$. If $\delta\epsilon = 0$, then $g = \pm 1$, $\delta = \epsilon = 0$. If $\delta\epsilon \neq 0$, $(g^2 - 9)^2(g^2 - 9/4) = 0$. If $g = \pm 3$, $\delta = -2$, $\epsilon = 2$. If $g = \pm 3/2$, $\delta = 1/4$, $\epsilon = -1/4$. The equations preceding (15) become

$$gp = gj = 1, \qquad gn = -2\epsilon, \qquad gk = 2\delta, \qquad g^3 m = \epsilon^2 + 2\delta,$$
$$g^3 h = 2\epsilon - \delta^2, \qquad 2g^3 c = 1 - g^2 - 2\delta\epsilon.$$

* The only singular point of $x^2y + t_3 = 0$ is (0, 1, 0, 0).

Changing the sign of g merely changes the signs of the elements of the first two rows of M. Hence we take $g = 1, 3, 3/2$ in turn and obtain

$$Y = Y' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix};$$

$$Z' = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & 0 \\ -\frac{4}{3} & \frac{2}{3} & -3 \\ 1 & 0 & 0 \end{bmatrix}, \quad W' = \begin{bmatrix} \frac{2}{3} & -\frac{4}{3} & 3 \\ \frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$M'' = \begin{bmatrix} \frac{1}{6}(-y+z-w), & \frac{1}{6}(-y+4z+2w), & \frac{3}{2}w \\ \frac{1}{6}(y+2z+4w), & \frac{1}{6}(y-z+w), & -\frac{3}{2}z \\ z & w & y \end{bmatrix}.$$

The conditions $YQ = RY'$, $ZQ = RZ'$, $WQ = RW'$ are satisfied if and only if

$$R = r\begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}, \quad Q = r\begin{bmatrix} -1 & 2 & -6 \\ 2 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix},$$

so that M and M' are equivalent. But in M'' the coefficient Y'' of y is of rank 2, while Y is of rank 1, so that M and M'' are not equivalent. Hence two matrices of type M with the same determinant may or may not be equivalent.

A second question relates to a possible simplification of our initial problem in advance of its solution. Let a and b be given, while the remaining parameters $c, h, \cdots, g$ in M are indeterminates. Can we find matrices R and Q with elements independent of $c, \cdots, g$ such that $MQ = RM'$ for suitably determined elements $c', \cdots, g', a, b$ of M'? If we can find such matrices R and Q not both similarity matrices rI and qI, we can employ them to normalize M formally in advance of its computation. Denote the elements of the ith row and jth column of R and Q by r_{ij} and q_{ij} respectively. By the third columns of $YQ = RY'$,

$$cq_{13} + hq_{23} \equiv r_{13}, \qquad mq_{13} - cq_{23} \equiv r_{23},$$

identically in c, h, m. Hence $q_{13} = q_{23} = r_{13} = r_{23} = 0$. By the third elements of the third rows of $ZQ = RZ'$ and $WQ = RW'$,

$$-ar_{31} - g'r_{32} = q_{13} = 0, \qquad g'r_{31} - br_{32} = q_{23} = 0.$$

By hypothesis, their determinant $ab + g'^2$ is not zero. Hence $r_{31} = r_{32} = 0$. Then by the third rows of our matrix products,

$$q_{11} = q_{22} = q_{33} = r_{33}, \qquad q_{ij} = 0 \quad (i \neq j), \qquad Q = r_{33}I.$$

Since we may replace R by $r_{33}^{-1}R$, we may take $r_{33} = 1$. We now employ matrix $P = R^{-1} = (p_{ij})$, which has $p_{13} = p_{23} = 0, p_{33} = 1$. Then $PM = M'$. In

$$PY = \begin{bmatrix} cp_{11} + mp_{12} & hp_{11} - cp_{21} & 0 \\ cp_{21} + mp_{22} & hp_{21} - cp_{22} & 0 \\ cp_{31} + mp_{32} & hp_{31} - cp_{32} & 1 \end{bmatrix} = Y',$$

we see by the last row and the sum of the diagonal elements that

$$p_{31} = p_{32} = p_{11} - p_{22} = p_{12} = p_{21} = 0.$$

Then $p_{11}^2 = 1$ by $|M| = |M'|$. For $p_{11} = +1$, we have the trivial case $P = Q = I$. For $p_{11} = -1$, the multiplication of M by P on the left is equivalent to changing the signs of the elements of the first two rows. But the elements a and b in Z and W are to remain unchanged. Hence *formal normalization of M is possible only when $a = b = 0$ and then consists in changing the signs of the elements of the first two rows.* Hence the equation for g then involves only even powers.